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The time-dependent motion due to a cylinder moving in an unbounded rotating or stratified fluid

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A rigid cylinder is initially at relative rest in a uniformly rotating, inviscid, incompressible fluid, with its generators perpendicular to the axis of rotation. The fluid is accelerated suddenly to a small constant velocity parallel to the axis of rotation, which is maintained thereafter. The growth of the subsequent disturbance due to the cylinder is interpreted in terms of plane inertial waves, the disturbance energy propagating with the local group velocity, which is in the plane of the wave front and proportional to the wavelength. Taylor columns, in which the fluid moves with the cylinder rather than round it, grow indefinitely in both directions parallel to the rotation axis, the head of the column moving with finite speed.

In a slightly viscous fluid, an ultimate steady state is reached, in which the columns are of finite length.

If a cylinder is moved horizontally in a non-rotating uniformly stratified Boussinesq liquid, an identical analysis may be applied with a similar interpretation in terms of internal gravity waves rather than inertial waves.

1. Introduction

This analysis is intended to throw light on the growth of Taylor columns in a uniformly rotating homogeneous liquid at low Rossby number, and also on the blocking by an obstacle of the two-dimensional motion of a stably stratified fluid at small internal Froude number. The analogy between two-dimensional motions in a fluid of non-uniform density and axisymmetric motions in a rotating fluid is well known (Yih 1965, ch. 6). There is also a close similarity between such axisymmetric flows and planar ones in which everything is independent of distance z in a direction which is fixed in a rotating frame of reference and perpendicular to the rotation vector.

In this case (figure 1) the components (u', v') of velocity in the (x, y)-plane, parallel and perpendicular to the rotation axis Ox respectively, correspond to the axial and radial velocities in the axisymmetric motion and are described by a Cartesian stream function ψ. The component $w'(x, y)$ in the Oz direction corresponds to a swirl velocity measured relative to solid body rotation. The Coriolis force, with components $(0, -2\Omega' w', 2\Omega' v')$, acts in the same manner in both systems. The centrifugal acceleration can be expressed as the gradient of a potential, and if the boundary conditions do not involve pressure explicitly it has no effect on the motion. The essential difference between the axisymmetric and the planar systems lies in the cylindrical as opposed to rectangular geometry. Experimentally, the former is much more convenient; analytically the latter is simpler, involving sines and cosines in lieu of Bessel functions. Dynamically, there is a close parallel between them.

However, there is correspondence between the planar motion in a rotating frame, and the strictly two-dimensional motion of a Boussinesq liquid in a vertical plane relative to a basic state in which Brunt–Väissälä frequency N is independent of position (Trustrum 1964). The role of the component $2\Omega' w'$ of Coriolis force is taken over by a buoyancy force per unit mass σ'. Changes of w' are equal to $-2\Omega'$ times the displacement η' of a fluid particle in the Oy direction; changes in σ' are equal to $-N^2\eta'$. In either case there is restoring force in the Oy direction everywhere proportional to $-\eta'$. This imparts a certain elasticity to the fluid motion, resulting in wavelike motions, known as inertial and internal gravity waves respectively. The main thesis of this paper is to relate the time-dependent growth of a region of stagnant fluid ($u' = v' = 0$, $w', \eta' \neq 0$) ahead of an obstacle placed in a uniform stream parallel to Ox to known properties of inertial and internal gravity waves. It is developed in the planar geometry, although similar things can be done in the axisymmetric system.

The verbal description is entirely in terms of rotating systems and inertial waves; an instant translation can be made to uniformly stratified systems and internal gravity waves. In the absence of viscosity and diffusivity for density, the mathematical analogy is exact, including non-linear terms. To maintain this with dissipation it is necessary that the viscosity be the same in both cases and equal to the diffusivity for density, and that on rigid boundaries the variations in density from the basic state vanish. However, we will be mainly concerned with non-dissipative fluids, except in §9, and even there the somewhat artificial boundary condition on the density plays no part in the theory and the results are easily generalized to unequal diffusivities.

Taylor (1922) observed that, when a sphere of radius a' is moved with speed U' parallel to the axis of rotation in a fluid which has angular velocity Ω', a long column of fluid is pushed ahead of the sphere with the same velocity, provided the Rossby number,

$$U = U'/2\Omega' a',$$

is less than about 0·15. This was confirmed for a streamlined sphere in a tube of radius $4a'$ by Long (1953), who also observed wavelike motions extending a great distance behind the body. No quantitative experimental estimates have yet been published of the length of the column, nor of the effect of the fluid viscosity on it.

Stewartson (1952) has analysed the disturbance produced by a sphere in an unbounded inviscid liquid which is initially in solid body rotation, when the fluid at large distances is suddenly set into motion with a constant infinitesimal velocity parallel to the axis of rotation. After a long time the velocity is almost everywhere independent of distance parallel to the rotation axis. A stagnant column extends ultimately indefinitely far ahead and behind the sphere. Stewartson (1958) has also shown that, if the Rossby number U is finite and equal to 1/5·76, no axisymmetric steadily translating solution exists for an inviscid fluid, in which the flow far upstream is merely solid-body rotation. From these two results he surmises that for all Rossby numbers less than a critical value which is at least $1/5{\cdot}76$, a column in which there is a substantial disturbance to uniform flow should in an inviscid fluid extend ultimately indefinitely far upstream.

A feature of his (1952) analysis, which assumes infinitesimal U, is that indefinitely large velocities and singularities build up in the solution, ultimately vitiating the linearization on which it is based. In the eventual steady state which is predicted by the theory these singularities extend all along the bounding surface of the stagnant column and on the body itself. However, if the Rossby number U is finite but sufficiently small, the development of the flow pattern may be followed for an arbitrary long time and to any given extent before the non-linear terms become large enough to invalidate the analysis. In this paper, we follow his approach closely, looking more carefully at the solution for large but finite t, in particular in regions far from the body where the disturbance has only just arrived, also near the edges of the Taylor column, and also close to the body. In all cases the development may be attributed to inertial waves. Far from the body the larger-scale waves arrive first, the frequency depending only on direction. In the growing Taylor column, the waves have zero frequency, but finite group velocity, which is parallel to the axis of rotation. The continuing input of energy into the column is associated with continuous emission of inertial waves from the body. The singularities at the edge of the column only build up as the smaller-scale waves (which travel more slowly) arrive. Near the surface of the body is a residual of very small-scale waves which have not yet had time to propagate away. This residual becomes a singularity because the length scales tend to zero while the velocity remains finite.

Following the insight provided by this interpretation, it is suggested that, provided the imposed Rossby number U is sufficiently small, a small non-zero kinematic viscosity ν' should limit the columns to a length of order $\Omega' a'^3/\nu'$, and an expression is given for the structure of such a viscous limited column. This is reminiscent of the wake far from a small sphere in a very viscous rotating fluid (Childress 1964). It is derived by remarking that inertial waves of smaller scale travel more slowly and are dissipated by viscosity more quickly. A wave of scale $\sqrt{(\nu'/2\Omega')}$ is dissipated before it has propagated more than a wavelength or so; whereas if $\sqrt{(\nu/2\Omega' a'^2)}$ is small, the waves of scale a' which are mainly responsible for the growth of the Taylor columns will travel a long way before the effects of viscosity become appreciable. Thus the column builds up substantially as in an inviscid fluid, but it is ultimately of finite length, and without singularities at the edge.

Two problems are treated in this paper, which are planar analogues of the one considered by Stewartson. We consider an infinite cylinder of circular cross-section with its axis perpendicular to the axis of rotation, and at rest in a rotating frame. In the first problem the fluid at infinity is moved impulsively an infinitesimal distance X' parallel to the rotation axis and then brought to rest again. The velocities are thereafter confined to an ever-expanding region around the cylinder. In the second problem the fluid is impulsively set into slow uniform maintained motion U', parallel to the rotation axis. In both problems it is consistent to assume that the flow is everywhere independent of position along the cylinder, just as it is independent of azimuthal angle in the axisymmetric problem. The natural analysis is then in terms of sines and cosines instead of Bessel functions, and the inertial waves generated are plane, instead of axisymmetric. However, the points of similarity between the solutions in these different geometries are numerous. The only qualitative difference which has been detected is the absence in the plane case of an analogue to the persistent oscillation along the axis of symmetry which was noted by Stewartson, and also by Sarma (1957), and the dynamics of the two situations are closely related.

As throughout this paper the fluid velocities are presumed infinitesimal, the equations of motion are linear and solutions may be superimposed. The solution to the second problem considered, in which the flow at large distances is uniform and maintained, may be found by the superposition of a large number of solutions of the first problem, in which there is a uniform impulsive displacement of large distances. The solution to the first problem is in some ways more easily understood, and is discussed in §§4 and 5, and the results for the other with their interpretation are given in §6. The analysis is quite straightforward, but tends to be somewhat lengthy because of the numerous different cases to be considered. For simplicity, the derivation of all these results is postponed to §§7 and 8, and may be omitted by the less conscientious reader.

One limitation of the analysis must be continually borne in mind. The linearized equations are entirely symmetrical between upstream and downstream. Thus for every Taylor column developing ahead of the cylinder there is an identical one found behind the cylinder. No explanation can appear within this framework of the observed asymmetries, e.g. the waves found in the wake by Long (1953, 1955).

2. The problems

A circular cylinder of radius a' has its axis Oz perpendicular to the basic rotation, which is Ω' about Ox (figure 1). It is consistent for the motion to be independent of the co-ordinate z. Using non-dimensional cartesian co-ordinates (x, y, z) based on length scale a' and using a time scale $(2\Omega')^{-1}$, we introduce a stream function $\psi(x, y, t)$, so that the velocity components are

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}, \quad w(x, y, t). \tag{2.1}$$

The momentum equations for an inviscid fluid,

$$\frac{Du}{Dt} + \frac{\partial\pi}{\partial x} = 0, \tag{2.2}$$

$$\frac{Dv}{Dt} - w + \frac{\partial\pi}{\partial y} = 0, \tag{2.3}$$

$$(Dw/Dt) + v = 0, \tag{2.4}$$

include the Coriolis force. That for motion along the cylinder may be integrated

$$w = -\int_0^t v\,dt, \tag{2.5}$$

where on the right-hand side the Lagrangian integral following a fluid particle is of course the lateral displacement (in the Oy direction) of the particle since the motion began. Thus the only way in which the motion in the Oxy-plane perpendicular to the cylinder axis is affected by rotation is through a restoring force w in the Oy direction on each fluid particle which is directly proportional to its displacement.

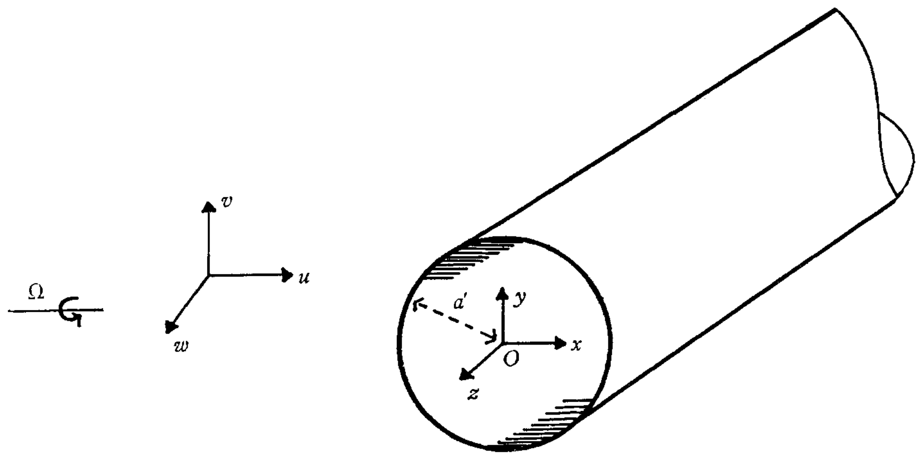


FIGURE 1. Planar motion round a circular cylinder.

Equations (2.1)–(2.5) are identical to those for the two-dimensional flow of a non-rotating Boussinesq liquid which is initially uniformly stratified, so that the density decreases linearly with height y through a very small range. The buoyancy force on an element of fluid is then directly proportional to its vertical displacement (in the Oy direction) since the motion began, and it appears in the vertical momentum equation exactly as w does in equation (2.3).

The boundary conditions are

$$\psi = 0, \quad w = 0, \quad \text{everywhere if} \quad t < 0, \tag{2.6}$$

$$\psi = 0 \quad \text{on} \quad r = \sqrt{(x^2+y^2)} = 1 \quad \text{if} \quad t > 0, \tag{2.7}$$

and, for the first problem

$$\psi = X\delta(t)y + o(r) \quad \text{as} \quad r \to \infty \quad \text{if} \quad t \geqslant 0; \tag{2.8}$$

or, for the second problem

$$\psi = Uy + o(r) \quad \text{as} \quad r \to \infty \quad \text{if} \quad t \geqslant 0. \tag{2.9}$$

The solution of equations (2.1)–(2.4) during the impulsive motion at $t = 0$, postulated in equation (2.8), may be obtained immediately. It is clearly irrotational, satisfying

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \quad \text{at} \quad t = 0,$$

but w increases according to equation (2.5) from zero up to a finite value which depends upon position and is proportional to X. Thereafter all velocities are finite, and if X is small compared to unity, the non-linear terms

$$u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}$$

in D/Dt are small compared to $\partial/\partial t$, and equations (2.2)–(2.4) may be linearized to give

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\frac{\partial^2\psi}{\partial x^2}=0. \tag{2.10}$$

In fact, during the impulsive displacement at $t = 0$, the accelerations are everywhere so large that the linearization is valid then also, and the boundary condition (2.8) may consistently be applied direct to equation (2.10). Alternatively if boundary condition (2.9) is applied instead, the velocities are to begin with of order the Rossby number, and provided this is small the equations may be linearized in the same way.

Equation (2.10) is solved subject to boundary conditions (2.6)–(2.8) or (2.6)–(2.9) by taking the Laplace transform $\hat{\psi}(x, y, s)$ with respect of time t, and obtaining a complete formal solution in terms of an integral in the complex plane of the transform variable s. The asymptotics as $t\to\infty$ for different regions in the (x, y)-plane are then found in terms of the behaviour of $\hat{\psi}(s)$ near its singularities. These are fairly straightforward to obtain, provided only the dominant term of the asymptotic expansion is required. Derivations are given in § 8.

3. Plane inertial waves

The properties of inertial waves in a uniformly rotating homogeneous fluid are derived succinctly in Chandrasekhar (1961, p. 85). If we write

$$x = r\cos\phi, \quad y = r\sin\phi,$$

the surfaces

$$r\sin(\phi-\theta) = y\cos\theta - x\sin\theta = \text{constant}$$

are planes, making an angle θ with the rotation axis Ox. There exist solutions of equation (2.10) of the form

$$\psi = R[A\exp i\{\alpha\, r\sin(\phi-\theta)-\sigma t\})]$$

which represent plane inertial waves, provided

$$\sigma = \pm\sin\theta. \tag{3.1}$$

The frequency thus depends only on the direction of the wavefronts. The group velocity, however, does depend on the wavelength $2\pi\alpha^{-1}$, and is parallel to the wave-front with magnitude $\cos\theta/\alpha$. The anisotropy of these waves, together with the fact that group velocity is parallel to the fluid motion and perpendicular to the phase velocity, accounts for almost all the novel features of the solutions obtained here. Two special cases are important. As $\theta\to\frac{1}{2}\pi$, the motion has a frequency equal to twice the rotation rate, and the fluid particles move in circles parallel to the Oyz-plane, but the group velocity vanishes. It will be seen that oscillations of this frequency do not propagate in the same way as the others. As $\theta \to 0$, the period of motion tends to infinity, and the length scale parallel to the axis of rotation becomes very much larger than the lateral scale. The group velocity is then finite and non-zero and parallel to Ox. Waves of this type account for the growth of the Taylor columns. The corresponding theory for internal gravity waves is developed in Phillips (1966).

4. The structure for large *t*

At $t = 0+$, immediately after the impulsive displacement, there are no velocities in the (x, y)-plane, $(\psi = 0)$, but the component w parallel to the cylinder does not vanish. After about one rotation period (t of order unity), the Coriolis force has induced again significant velocities (u, v), but these are rotational, and the

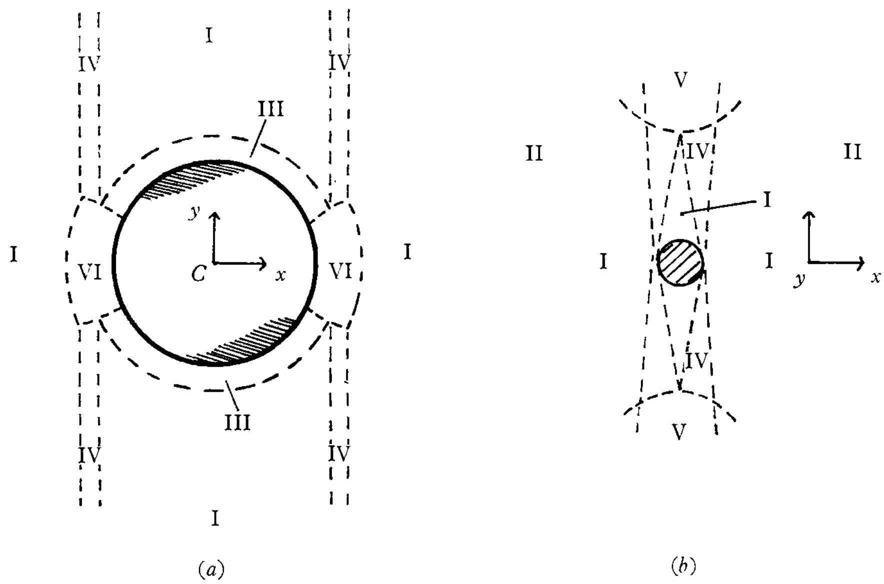


FIGURE 2. Asymptotic regions after an impulsive displacement, (*a*) near the cylinder, (*b*) at large distances.

flow field bears no resemblance to that during the displacement. After many rotation periods $t \gg 1$, a distinct structure reappears. The (x, y)-plane may then be divided into six distinct regions, in each of which there is a different asymptotic form for the solution as $t \to \infty$. They are sketched in figure 2.

In region I, which arises almost everywhere when the point $P(x, y)$ under consideration is held fixed as $t \to \infty$, the dominant motion in the first problem is the sum of two trains of locally plane inertial waves of different frequencies, of which the wavelength and amplitude decreases steadily to zero. The source regions for these waves may be identified with the points of contact T_1, T_2 of the tangents through P to the cylinder (figure 3).

In the second problem, in which the velocity at infinity is maintained after the impulsive start, the motion at any given point does not die out with time. Over the whole of region I there appears a steady velocity field (including the Taylor column), on which are superposed decaying trains of inertial waves very similar to those for the first problem.

Region II lies outside region I, and is obtained by holding $r/t = \sqrt{(x^2+y^2)}/t$ fixed as $t \to \infty$. Thus a point P in region II moves steadily outwards from the cylinder with constant velocity. The motion is a pattern of modulated inertial waves of wavelength comparable to the radius of the cylinder, travelling anisotropically outwards with P. The sources T_1 and T_2 can no longer be regarded as distinct, but form a single region of origin for the wave.

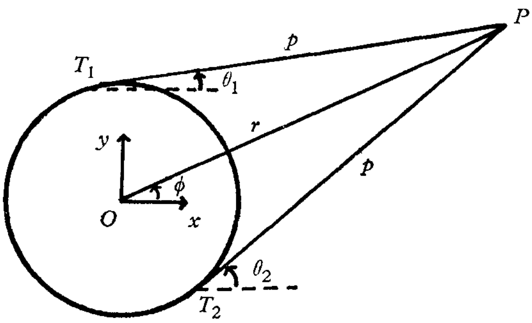


FIGURE 3

Region III forms a ring around the cylinder, and decreases in thickness like t^{-2} as t increases. The velocities in region III do not decay with time; there is always an oscillation of finite, non-zero magnitude on the surface of the cylinder. As the thickness of this region shrinks to vanishing point, it appears as a singularity in the solution for region I. It may be ascribed to the remnant of inertial waves excited at time $t = 0$, but of such small length scale and group velocity, that they have not yet had time to propagate away from their neighbourhood of origin.

The asymptotic structure in each of regions I, II and III fails along the lines $x = \pm 1$, which are perpendicular to the axis of rotation and tangent to the cylinder. At such points the inertial waves predicted have frequency twice the basic rotation rate, and zero group velocity. The analytical solution in these regions IV, V, VI is not easily expressed in terms of tabulated functions and is omitted in this paper but its structure is clear. The velocity is everywhere nearly parallel to Oy, and near the cylinder in region VI has the functional form as $t \to \infty$

$$v \sim \frac{1}{\sqrt{t}} F(y\sqrt{t}, (x \pm 1)t) \begin{matrix}\sin\\ \cos\end{matrix} t \tag{4.1}$$

and far away in region V

$$v \sim \frac{1}{\sqrt{t}} G\left(\frac{y}{\sqrt{t}}, x\right) \begin{matrix}\sin\\ \cos\end{matrix} t, \tag{4.2}$$

whereas in between in region IV, where $y/\sqrt{t}$ is small but $y\sqrt{t}$ is, nevertheless, large,

$$v \sim \frac{1}{\sqrt{(y\sqrt{t})}} \frac{1}{\sqrt{t}} H\left(\frac{(x \pm 1)\sqrt{t}}{y}\right) \begin{matrix}\sin\\ \cos\end{matrix} t. \tag{4.3}$$

Although the dependence on $y\sqrt{t}$ and $y/\sqrt{t}$ in these formulae is suggestive of a diffusive process, the dynamics of these regions is still obscure. However, as $t \to \infty$, they occupy a smaller and smaller proportion of the (x, y)-plane, and the oscillations in them are generally weaker than in regions I, II and III at the same distance from the cylinder.

In the second problem additional singularities appear in the solutions for each of regions I, II, III along the lines $y = \pm 1$ which mark the edges of the Taylor column. The flow in these neighbourhoods is not really singular, but a different asymptotic representation is required. Thus three more regions VII, VIII, IX must be distinguished, corresponding respectively to the overlap with the first three.

All these regions merge into one another, in those places where the respective asymptotic expansions have common areas of validity. They only become distinguishable for t substantially larger than unity, and distinct only in the mathematical limit $t \to \infty$.

5. Results and discussion for the impulsive start (problem 1)

(*a*) *Region I*

To describe the solution in region I, we let θ_1 and θ_2 be the angles (between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$) made with the Ox axis by the tangents PT_1, PT_2 from the point $P(x, y)$ to the cylinder (figure 3). The lengths PT_1 and PT_2 are equal, being

$$p = \sqrt{(r^2 - 1)}.$$

The dominant contribution to the motion in region I is made up of two similar terms, associated with T_1 and T_2, of which the first is

$$\psi = \pm \frac{X}{\sqrt{(2\pi)}} \frac{1}{t} \sqrt{\left(\frac{p}{t \cos \theta_1}\right)} \cos(\theta_1 \pm \tfrac{1}{4}\pi + t \sin \theta_1). \tag{5.1}$$

For the determination of which of the $\pm$ signs is applicable, the reader is referred to § 8.

Equation (5.1) appears complicated, but for large t the dominant feature is the appearance of $t \sin \theta_1$ in the argument of the final cos. An infinitesimal displacement δq perpendicular to the tangent PT_1 gives rise to a change

$$\frac{t \cos \theta_1}{p} \delta q$$

in the value of $t \sin \theta_1$, whereas a displacement δp parallel to the tangent does not alter $t \sin \theta_1$, and is associated with a change in ψ which is smaller by $O(t^{-1})$. Thus equation (5.1) represents *modulated plane waves*, with wave fronts parallel to the tangent PT_1, with wave-number $t \cos \theta_1 / p$ perpendicular to the tangent, and frequency at a given point equal to $\sin \theta_1$. The remaining detail in expression (5.1) gives the amplitude and phase of the waves as a function of position. The magnitude of the velocity is found by differentiating ψ in the direction normal to the plane of the waves, and is

$$\frac{X}{\sqrt{(2\pi)}} \frac{\cos \theta_1}{p} \sqrt{\left(\frac{p}{t \cos \theta_1}\right)}. \tag{5.2}$$

For fixed p, θ_1 this decays like $t^{-\frac{1}{2}}$ so the oscillations eventually die out all over region I.

The local wave-number

$$\alpha = t\cos\theta_1/p$$

is constant at points which move along the tangent PT_1 away from T_1 with speed $\cos\theta_1/\alpha$, i.e. with *group velocity* of the local waves. Simultaneously the velocity amplitude falls inversely as the distance, so the kinetic energy associated with a given wave-number band α to $\alpha+\delta\alpha$ and a given sectorial angle θ_1 to $\theta_1+\delta\theta_1$ remains constant. The distribution of energy between these wave-numbers, namely

$$\frac{X^2}{2\pi}\frac{\cos^2\theta_1}{\alpha^2}\delta\alpha\,\delta\theta_1,$$

is a property of the source region for the waves near T_1.

Also found distributed over region I is an inertial oscillation with the cut-off frequency, which in dimensionless terms is unity;

$$\psi = -\frac{2X}{\sqrt{(2\pi)}}\frac{y}{\sqrt{t^3}}\left\{1-\frac{|x|}{\sqrt{(x^2-1)}}\right\}\cos(t-\tfrac{1}{4}\pi) \quad \text{if} \quad |x|>1,$$

$$\psi = -\frac{2X}{\sqrt{(2\pi)}}\frac{1}{\sqrt{t^3}}\{y-\sqrt{(1-x^2)}\operatorname{sgn} y\}\cos(t-\tfrac{1}{4}\pi) \quad \text{if} \quad |x|<1. \tag{5.3}$$

This is of smaller magnitude than the waves radiating from T_1, T_2. It may be verified that the v and w components of velocity have the same magnitude, but are $\frac{1}{2}\pi$ out of phase, so that the velocity vector in the Oyz-plane is circularly polarized in the opposite direction to the rotation. This is to be expected if the fundamental balance in equations (2.2) and (2.4) is between the accelerations and coriolis forces. However, the motion is distributed over the whole of space, and cannot be described as plane waves. The interpretation of the development of motions of this frequency (for which the group velocity vanishes) is not entirely satisfactory.

(*b*) *Region II*

If at given finite t, r is taken sufficiently large, the angles θ_1, θ_2 made with Ox by the tangents from P to the cylinder nearly coincide. It is then not correct to describe the inertial waves produced by the cylinder as the sum of two terms like equation (5.1), each associated with a distinct source near T_1 and T_2 respectively. Instead, if

$$x = r\cos\phi \quad y = r\sin\phi,$$

away from the direction $\phi = \pm\frac{1}{2}\pi$ the dominant part of the velocity field is

$$\psi \sim \mp\frac{X}{t}J_1\left(\frac{t\cos\phi}{r}\right)\sin(\phi+t\sin\phi) \quad \text{as} \quad t\to\infty. \tag{5.4}$$

In this and subsequent expressions ϕ is taken to lie between 0 and $\frac{1}{2}\pi$. The value for other quadrants may be obtained by symmetry. Equation (5.4) describes a single system of modulated plane waves. The wave-fronts are in the radial direction ϕ = constant, whereas the wave-number

$$\alpha = |\nabla(t\sin\phi)| = \frac{t\cos\phi}{r},$$

and the local frequency is $\sin\phi$. A given wave-number α is associated with a velocity of amplitude

$$X\frac{\cos\phi}{r}J_1(\alpha),$$

where $J_1(\alpha)$ is the Bessel function of order 1, and the point at which a given wave-number is found moves radially with speed $\cos\phi/\alpha$ which is again the appropriate group velocity from plane wave theory. The zeros of $J_1(\alpha)$ imply bands of quiescent fluid, which expand outwards with the general pattern (figure 4). The wave-fronts all appear to move round circles r = constant towards the rotation axis.

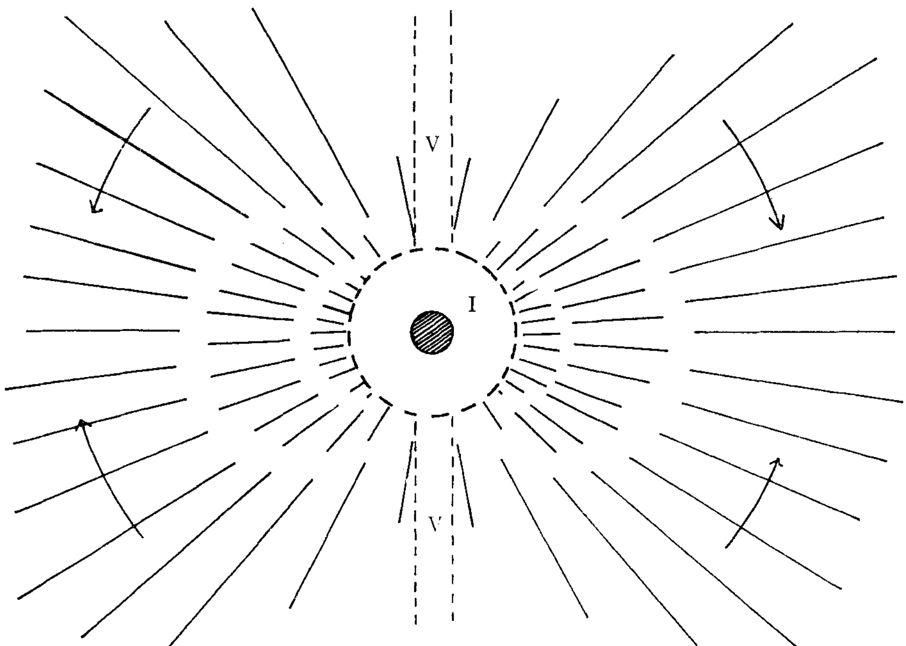


FIGURE 4. The wave pattern in region II at large distances from the cylinder caused by an impulsive displacement.

At any finite time, at sufficient distance from the cylinder $\alpha = t\cos\phi/r$ is small, and the velocity amplitude is

$$Xt\frac{\cos^2\phi}{2r^2},$$

which decreases rapidly with distance. If, on the other hand, $\alpha = t\cos\phi/r$ is large, there is a region of overlap with region I. On replacing $J_1(\alpha)$ by the leading term in its expansion for large α, equation (5.4) may be written as the sum of two terms like equation (5.1). Thus region II embraces all those places where the local wavelength $2\pi\alpha^{-1}$ is comparable with the cylinder diameter or larger, but it goes over to region I where the wavelength is substantially shorter. In region I, the neighbourhoods of T_1 and T_2 may be clearly distinguished as source regions for the waves. In region II the whole area around the cylinder must be taken as the source.

As $\phi \to \pm\frac{1}{2}\pi$, the velocities predicted by equations (5.4) vanish, so a different asymptotic expression is required. Equation (4.2), describing region V has the appropriate structure.

Although the largest velocity amplitudes are found in the inner edge of region I, where the local wave-number α is very large, the energy radiated by the source is to be found predominantly in region II, carried by the waves of scale comparable to that of the cylinder. The energy (all kinetic) between wave-numbers α to $\alpha+\delta\alpha$ in the sectorial angle ϕ to $\phi+\delta\phi$ is

$$X^2 \cos^2\phi (1/\alpha) J_1^2(\alpha)\, \delta\alpha\, \delta\phi. \tag{5.5}$$

The integral of this converges, and the total radiated energy is

$$\int_0^{2\pi} d\phi \int_0^{\infty} d\alpha\, X^2 \cos^2\phi (1/\alpha) J_1^2(\alpha) = \tfrac{1}{2}\pi X^2. \tag{5.6}$$

(c) *Region III*

Very near the cylinder, the stream function for large values of t is

$$\psi \sim \pm X(\sqrt{(2r-2)}/t) J_1(t \sin\phi \sqrt{(2r-2)}) \sin(\phi - t\cos\phi). \tag{5.7}$$

The length of tangent

$$p = \sqrt{(r^2-1)} \sim \sqrt{(2r-2)}$$

is still important in the dynamics, but the radial distance from P to the cylinder is proportional to p^2, and equation (5.7) has been put in a form which brings this out. On the cylinder ψ vanishes, but the tangential velocity is

$$\partial\psi/\partial r = \pm X \sin\phi \sin(\phi - t\cos\phi) \quad \text{on} \quad r = 1. \tag{5.8}$$

There is thus a finite oscillation on the cylinder, which does not decay as $t \to \infty$. However, this velocity only extends through the region where $t\sin\phi\sqrt{(2r-2)}$ is small compared to unity. The layer of maximum velocity is of thickness of order $(t\sin\phi)^{-2}$, and shrinks rapidly as t increases. The length scale around the cylinder is of order $(t\sin\phi)^{-1}$ throughout region III, and hence the motion is predominantly parallel to the local tangent plane.

This region of finite amplitude oscillation may probably be described as the residual of inertial waves formed at time $t = 0$ of such short wavelengths that they have not yet had time to propagate away from the source region. At the outer edge of region III, the two separate propagating wave-trains associated with source regions T_1 and T_2 can already be identified. As t increases shorter and shorter waves become discernable at this outer edge.

The dominant wave-number α there is almost in the radial direction, and may be computed from the assumption that the wave has travelled in time t a distance $p \sim \sqrt{(2r-2)}$ from T_1 or T_2, with group velocity $\sin\phi/\alpha$,

$$\alpha = \frac{t\sin\phi}{\sqrt{(2r-2)}}.$$

Such a wave will only be distinguishable if it is a substantial number of wavelengths from the cylinder, i.e. if

$$(r-1)\alpha = \tfrac{1}{2}\sqrt{(2r-2)}\, t\sin\phi \gg 1.$$

Nearer the cylinder, the wave dispersion has not yet had time to establish a dominant length scale, and the radial structure described by equation (5.7) is not sinusoidal. However, the motion is approximately parallel to the local tangent plane, and has the appropriate frequency.

6. Results and discussion (problem 2)

(*a*) *Region I*

When the imposed velocity of the fluid at infinity is maintained after the impulsive start, the solution obtained in §7 is formally the integral from 0 to t of the solution for the impulsive displacement. The features found when t is large can be classified, on the one hand as decaying oscillating residuals generated in the impulsive start, or on the other hand as constant or growing effects due to the systematic build-up of continuously generated wave-motions of small or zero frequency.

The oscillatory motions in region I are very similar to those with the impulsive start. There are plane propagating inertial waves originating near T_1 and T_2,

$$\psi = \mp\frac{U}{\sqrt{(2\pi)}}\frac{1}{t\sin\theta_1}\sqrt{\left(\frac{p}{t\cos\theta_1}\right)}\sin\left(\theta_1 \pm \tfrac{1}{4}\pi + t\sin\theta_1\right), \tag{6.1}$$

together with a formally identical contribution involving θ_2.

There is also an inertial oscillation (not propagating)

$$\begin{aligned}\psi &= -\frac{2U}{\sqrt{(2\pi)}}\frac{y}{\sqrt{(t^3)}}\left\{1-\frac{|x|}{\sqrt{(x^2-1)}}\right\}\sin\left(t-\tfrac{1}{4}\pi\right) \quad \text{if} \quad |x|>1,\\ &= -\frac{2U}{\sqrt{(2\pi)}}\frac{1}{\sqrt{(t^3)}}\left\{y-\sqrt{(1-x^2)}\,\mathrm{sgn}\,y\right\}\sin\left(t-\tfrac{1}{4}\pi\right) \quad \text{if} \quad |x|<1.\end{aligned} \tag{6.2}$$

These have a phase-shift from those of the first problem, and $\sin\theta_1$ appears in the denominator of equation (6.1), but they are otherwise identical. This factor implies an apparent singularity along $y = \pm 1$ (for the resolution see region VII).

These oscillatory motions are superposed on a much larger steady flow (the Taylor column), given by

$$\left.\begin{aligned}\psi &\to U\,\mathrm{sgn}\,y\sqrt{(y^2-1)} && \text{if} \quad |y|>1\\ &\to 0 && \text{if} \quad |y|<1\end{aligned}\right\}. \tag{6.3}$$

From this it appears that the ultimate steady state is everywhere independent of distance along the rotation axis, and the transverse velocity v vanishes. Within the Taylor column $|y| < 1$, the longitudinal velocity u also vanishes, and the flow is completely blocked by the cylinder, whereas outside the column it is given by

$$u = U\frac{|y|}{\sqrt{(y^2-1)}}. \tag{6.4}$$

Thus only a few diameters to either side of the cylinder, the longitudinal velocity u is very close to the uniform value U imposed at infinity. The singularity at

$y = \pm 1$ is only apparent, for at any finite time such a point is always within region VII. The velocity parallel to the cylinder is given as $t \to \infty$ by

$$\begin{aligned} w &\to 0 \quad \text{if} \quad |y| > 1 \\ &\to U \frac{y}{\sqrt{(1-y^2)}} \operatorname{sgn} x \quad \text{if} \quad |y| < 1. \end{aligned} \tag{6.5}$$

Because $-w$ is a measure of the displacement in the positive Oy direction (vertical) since the motion began, it is seen that upstream of the cylinder all the particles in the column have been displaced away from the Ox axis, whereas downstream they have been displaced towards it. This is in the direction to be expected intuitively. Outside the column the ultimate displacement must be quite independent of x, and hence, by symmetry about $x = 0$, it vanishes.

(*b*) *Regions II and III*

As $r \to \infty$ with t, or as $r \to 1$, the expressions given for region I need modification. Except in the direction of the axis of rotation ($\phi = 0$ or π), the perturbation from a uniform stream steady velocity of equation (6.4) decreases like r^{-2} at large distances, and is smaller than the plane inertial waves (6.1). The largest two terms in the velocity field are, for $0 < \phi < \frac{1}{2}\pi$,

$$\psi \sim Uy + \frac{U}{t \sin \phi} J_1\left(\frac{t \cos \phi}{r}\right) \cos(\phi + t \sin \phi). \tag{6.6}$$

Noteworthy is the appearance of $\sin \phi$ in the denominator, so this expression cannot hold in the direction $\phi = 0$. The interpretation of these terms is the same as for problem 1.

If $r = 1 + O(t^{-1})$ as $t \to \infty$, region III is appropriate, and

$$\psi \sim U \frac{\sqrt{2(r-1)}}{t \cos \phi} J_1(t \sin \phi \sqrt{2(r-1)}) \cos(\phi - t \cos \phi). \tag{6.7}$$

This time $\cos \phi$ appears in the denominator.

(*c*) *Regions VII and VIII*

The frequency of the inertial waves which propagate along the rotation axis vanishes. If a source of such waves is maintained, waves emitted at different instants of time, each with the same phase at the moment of emission, will not destructively interfere but will build up. Thus, in these directions a linear growth in the magnitude of the velocity is to be anticipated. However, destructive interference may occur between waves of different wave-numbers, for although they propagate at different speeds a full spectrum radiated from a maintained source will ultimately be observed at any given point. At any finite time, there will be a cut-off wave-number above which no waves have yet reached the point under consideration. As more waves arrive the velocity field settles down to a steady value, except in certain places where the (spatial) phase of all the waves is the same, and the velocity goes on increasing indefinitely.

These considerations are completely adequate to account for the growth of the Taylor columns. The velocity components in the developing column (region VIII, see figure 5) are

$$\left.\begin{aligned} u &\sim U - U\int_0^{t/r} J_1(\alpha)\cos\alpha y\,d\alpha,\\ v &\sim -\operatorname{sgn} x\,\frac{U}{r}\,J_1\!\left(\frac{t}{r}\right)\sin\frac{yt}{r},\\ w &\sim \operatorname{sgn} x\,U\int_0^{t/r} J_1(\alpha)\sin\alpha y\,d\alpha.\end{aligned}\right\} \tag{6.8}$$

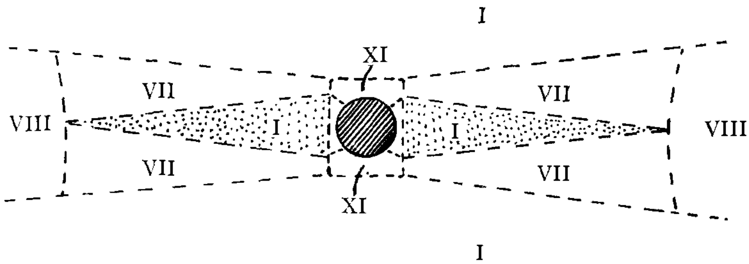


FIGURE 5. Additional regions for problem 2. The fully developed Taylor column is shown stippled.

Here, to a good approximation, $r = |x|$. A sketch of the velocity profile across the column for two different values of t/r is given in figure 6. Using known expressions for the infinite integral, equations (6.8) reduce as $t/r \to \infty$ to equations (6.4) and (6.5) which describe the ultimate steady state. When $t/r \to 0$ the disturbance velocities induced by the cylinder decrease like $(t/r)^2$. For given t/r and large $|y|$ these velocities match those in region II. Thus equation (6.8) describes without singularities the development of a Taylor column across its whole breadth.

The build up of a Taylor column is thus pictured as due to the arrival of the point under consideration of waves of successively smaller wavelength, filling in the details of the structure. The main feature is a velocity defect of order U, imparted by waves of length of order the radius of the cylinder. If the end of the column is defined to be where the velocity at $y = 0$ is (say) $\frac{1}{2}U$, the end of the column moves with constant speed given by

$$\int_0^{t/r} J_1(\alpha)\,d\alpha = \tfrac{1}{2}. \tag{6.9}$$

This means $r/t \sim 0{\cdot}67$, which is very much faster than the fluid velocity and is the same as the group velocity of an inertial wave of half wavelength almost equal to the diameter of the cylinder. As $t/r \to \infty$ the integrals (6.8) diverge at $y = \pm 1$. There all wavelengths combine, and the velocities u, w build up as $O(t/r)^{\frac{1}{2}}$ over a region of width $y = \pm 1 + O(r/t)$. Thus singularities develop along the edge of the column.

To complete the picture it is necessary to consider region VII near $y = \pm 1$, for given values of x as $t \to \infty$. Then near $y = 1$,

$$u \sim \frac{U}{\sqrt{(2\pi)}}\int_0^{t/|x|} \frac{1}{\sqrt{\alpha}}\sin\{(y-1)\alpha + \tfrac{1}{4}\pi\}\,d\alpha + u_2, \tag{6.10}$$

where u_2 is an oscillatory inertial wave emitted from source region T_2 according to equation (6.1). Equation (6.10) emphasizes that it is the distance $|x|$ of P from the point of contact T_1 of the tangent at $x = 0$, $y = 1$ which is the controlling factor, and also that, for any given t, the maximum velocities and the most

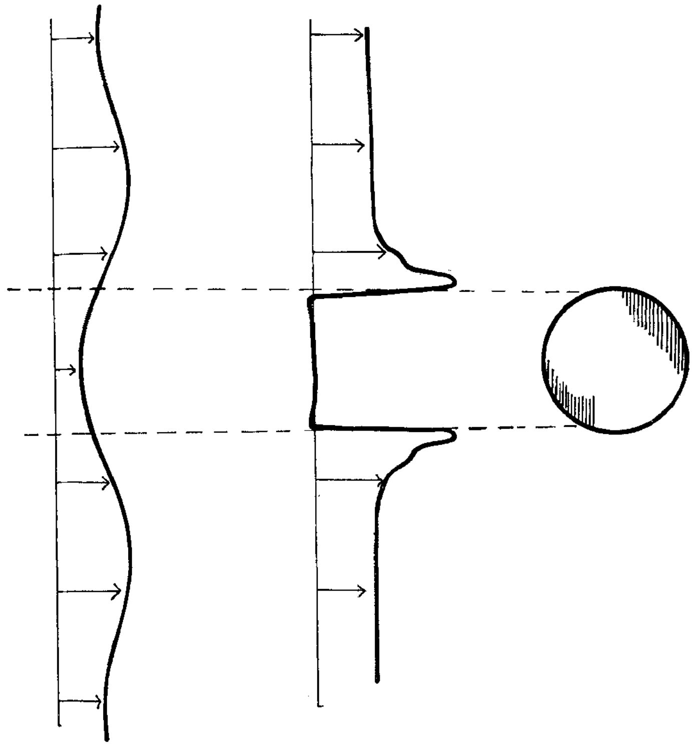


FIGURE 6. The velocity profile across a developing Taylor column. (*a*) $t/r = 1{\cdot}5$. (*b*) $t/r = 15$.

singular behaviour are not to be found in the Taylor column, but in region IX, near the cylinder. For computational purposes equation (6.10) may be put in the form

$$u \sim \frac{U}{\sqrt{(2|y-1|)}}\left\{C\left(\frac{t|y-1|}{|x|}\right)+\operatorname{sgn}(y-1)\,S\left(\frac{t|y-1|}{|x|}\right)\right\}+O(y-1),$$

where C and S are the Fresnel functions. This shows that the maximum value of u occurs when $y-1 = 1{\cdot}0\,(|x|/t)$, and is $0{\cdot}7\sqrt{(t/|x|)}$.

(*d*) *Region IX*

Here
$$\psi \sim U\sqrt{(2r-2)}\int_0^{t\sqrt{(2r-2)}} \frac{1}{\alpha} J_1(\alpha)\cos\frac{\alpha x}{\sqrt{(2r-2)}}\,d\alpha. \tag{6.11}$$

The thickness and structure of this region is very close to that of region III (equation (6.7)). However, because the local inertial frequency vanishes, the velocities induced by a continuous succession of displacements of the cylinder ultimately build up indefinitely (at least on a linearized theory). At any given time the maximum velocity is of order t, and the scale of variation in the direction of the velocity is of order t^{-1}. Hence the neglected non-linear terms $u(\partial/\partial x)$ increase like t^2, whereas the time derivative $\partial/\partial t$ decreases like t^{-1}. Thus, after a time of order $U^{-\frac{1}{3}}$, the linearization breaks down in a localized region. However, by that stage the main features of the flow pattern elsewhere have already taken on their distinctive character, and are unlikely to be influenced greatly by further propagation outwards of non-linear effects.

7. The formal solution

(a) The Laplace transform

Writing

$$\hat{\psi}(x,y,s) = \int_0^\infty \psi(x,y,t)\,e^{-st}ds, \quad R(s) > 0,$$

it may be verified that the solution to equation (2.10) subject to boundary conditions (2.6), (2.7), (2.8) for the first problem is

$$\hat{\psi} = \frac{X}{2\{s-(s^2+1)^{\frac{1}{2}}\}}[2sy - \{((s^2+1)^{\frac{1}{2}}y+isx)^2-1\}^{\frac{1}{2}} - \{((s^2+1)^{\frac{1}{2}}y-isx)^2-1\}^{\frac{1}{2}}], \tag{7.1}$$

where $(s^2+1)^{\frac{1}{2}} > 0$ for s on the positive real axis, and

$$\{((s^2+1)^{\frac{1}{2}}y+isx)^2-1\}^{\frac{1}{2}} \sim (s^2+1)^{\frac{1}{2}}y+isx,$$
$$\{((s^2+1)^{\frac{1}{2}}y-isx)^2-1\}^{\frac{1}{2}} \sim (s^2+1)^{\frac{1}{2}}y-isx.$$

as $r = \sqrt{(x^2+y^2)} \to \infty$ for given s. For this value of $\hat{\psi}$ satisfies;

(i) as $r \to \infty$ for given s, $\hat{\psi} \sim Xy$, which is the Laplace transform of equation (2.8);

(ii) on $r = 1$, $x = \cos\phi$, $y = \sin\phi$, and

$$\{((s^2+1)^{\frac{1}{2}}y+isx)^2-1\}^{\frac{1}{2}} = s\sin\phi + i(s^2+1)^{\frac{1}{2}}\cos\phi$$

so $\hat{\psi}(s) = 0$;

(iii) for real positive s,

$$\hat{\psi} = \frac{X}{\{s-\sqrt{(s^2+1)}\}} R\left[\frac{s}{\sqrt{(s^2+1)}}\{\sqrt{(s^2+1)}y+isx\} - \{(\sqrt{(s^2+1)}y+isx)^2-1\}^{\frac{1}{2}}\right],$$

which is throughout $r > 1$ an analytic function of $\sqrt{(s^2+1)}y+isx$. Hence for real positive s

$$(s^2+1)\frac{\partial^2\hat{\psi}}{\partial x^2} + s^2\frac{\partial^2\hat{\psi}}{\partial y^2} = 0, \tag{7.2}$$

which is the Laplace transform of equation (2.9). But throughout $\mathscr{R}(s) > 0$, $\hat{\psi}(s)$ is an analytic function of s if $r \geqslant 1$, so it also satisfies equation (7.2).

For the second problem boundary condition (2.9) must be taken. It is easily seen that in the expression (7.1) for $\hat{\psi}$, X must be replaced by U/s. Otherwise the two solutions are identical.

(*b*) *The inverse transform*

The complete formal solution to our problem is now

$$\psi(x,y,t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \hat{\psi}(s)\, e^{st} ds,$$
$$w(x,y,t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{1}{s}\frac{\partial\hat{\psi}}{\partial x}\, e^{st} ds,$$

where the integral is taken in the first instance in $\mathscr{R}(s) > 0$.

(*c*) *First change of variable*

Before discussing the behaviour of ψ it is convenient to make a change of variable which eliminates one of the branch cuts inherent in $\hat{\psi}(s)$.

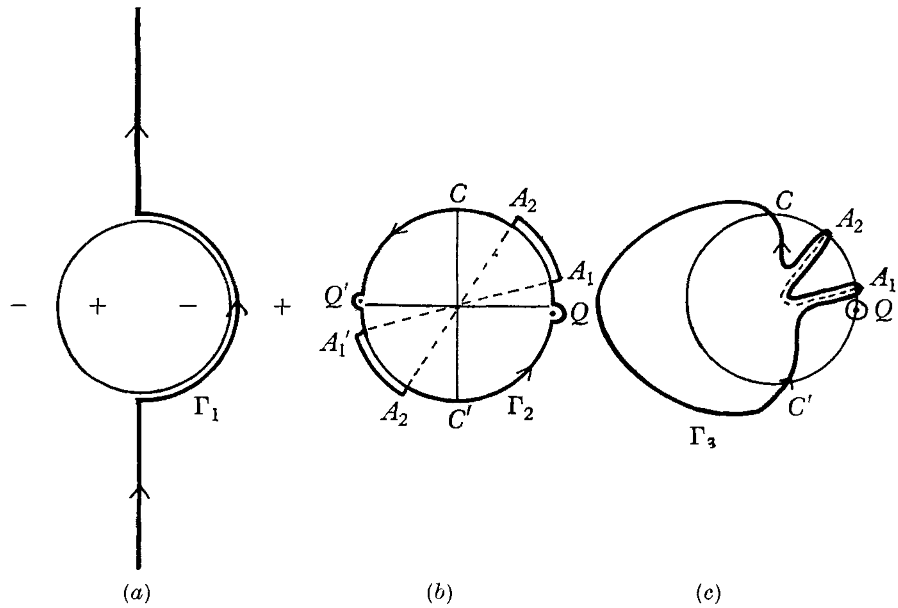


FIGURE 7. Contours of integration in the $(\zeta = \rho e^{i\theta})$-plane. The signs of $R(s)$ are indicated in (*a*).

We put

$$s = \tfrac{1}{2}(\zeta - \zeta^{-1}), \quad (s^2+1)^{\frac{1}{2}} = \tfrac{1}{2}(\zeta + \zeta^{-1}),$$

so that each point in the s-plane corresponds to two points on the ζ-plane, but the correct branch of $(s^2+1)^{\frac{1}{2}}$ is automatically chosen if the mapping is on to the region $|\zeta| \geqslant 1$. The imaginary axis in the s-plane, with the appropriate branch, becomes the contour Γ_1, in figure 7 (*a*). Extending $\hat{\psi}(\zeta)$ by analytic continuation into the whole ζ-plane, all the singularities are found on the circle $\rho = |\zeta| = 1$, and the contour of integration may be deformed into either of the forms shown in figures 7 (*b*), 7 (*c*).

Writing

$$\zeta = \rho e^{i\theta}$$

and considering the integral round the unit circle $\rho = 1$ of figure 7 (*b*), we have for problem 1,

$$\psi = \mathscr{R}\left[-\frac{X}{2\pi}\int_{-\pi}^{\pi}\cos\theta\exp\{i(\theta+t\sin\theta)\}\left[iy\sin\theta-\{(y\cos\theta-x\sin\theta)^2-1\}^{\frac{1}{2}}\right]d\theta\right], \tag{7.3}$$

$$w = \mathscr{R}\left[\frac{X}{2\pi}\int_{-\pi}^{\pi}\cos\theta\exp\{i(\theta+t\sin\theta)\}\left[-\frac{y\cos\theta-x\sin\theta}{\{(y\cos\theta-x\sin\theta)^2-1\}^{\frac{1}{2}}}\right]d\theta\right], \tag{7.4}$$

where the integrals are to be taken round singularities in the direction $\mathscr{I}(\theta) < 0$.

For problem 2,

$$\psi = \mathscr{R}\left[\frac{U}{2\pi}\int_{-\pi}^{\pi} i\cot\theta\exp\{i(\theta+t\sin\theta)\}\left[iy\sin\theta-\{(y\cos\theta-x\sin\theta)^2-1\}^{\frac{1}{2}}\right]d\theta\right]. \tag{7.5}$$

(*d*) *The singularities of* $\hat{\psi}$

The branch points of $\hat{\psi}(s)$ at $s = \pm i$ have been eliminated in the transformation to equation (7.3), but are replaced by points of stationary phase in the exponent $\exp(it\sin\theta)$ at $\theta = \pm\frac{1}{2}\pi$. These are denoted by C, C' in figure 7 (*b*). There are also branch points on the unit circle at the values of θ for which

$$y\cos\theta - x\sin\theta = \pm 1.$$

These points (A_1, A_2, A_1', A_2' in figure 7 (*b*)) may be characterized by the angles θ_1, θ_2 made by the two tangents PT, PT_2 (figure 3) from the point P, $(x, y) = (r\cos\phi, r\sin\phi)$ to the cylinder with the Ox axis, together with the reverse directions $\pi+\theta_1$, $\pi+\theta_2$. Without loss of generality we will take

$$-\tfrac{1}{2}\pi < \theta_1, \theta_2 \leqslant \tfrac{1}{2}\pi.$$

The quantities
$$\xi = x\cos\theta + y\sin\theta = \rho\cos(\phi-\theta),$$
$$\eta = y\cos\theta - x\sin\theta = \rho\sin(\phi-\theta), \tag{7.6}$$

which are the co-ordinates of the point P, (x, y) relative to axes rotated through an angle θ, are important variables in this analysis. At the point A_1,

$$\eta_1 = 1, \quad \xi_1 = \sqrt{(r^2-1)} = p,$$

where the positive signs are taken throughout the region $x > -1$, $y > 0$. At A_2,

$$\eta_2 = \mp 1, \quad \xi_2 = \pm p,$$

where the top signs are taken in $x > 1$, $y > 0$, but the signs change abruptly as x passes through the value 1. $p = \sqrt{(r^2-1)}$ is the length of the tangent PT.

Finally, in the second problem only, there is a pole Q at $\theta = 0$, corresponding to $s = 0$. This is a consequence of boundary condition (2.9), and expresses the fact that the flow at infinity is maintained steadily as $t \to \infty$.

(*e*) *The choice of branch*

The branch of
$$\Phi = \{[(s^2+1)y + isx]^2 - 1\}^{\frac{1}{2}}$$
$$= [\{\tfrac{1}{2}(\rho+\rho^{-1})\eta + \tfrac{1}{2}i(\rho-\rho^{-1})\xi\}^2 - 1]^{\frac{1}{2}},$$

has been defined only for s real and positive, as $r \to \infty$. We need it for given x, y

and θ as $\rho \to 1$ from above. It is an analytic function of ρ, θ, x, y everywhere in $\rho > 1, r > 1$. For $\rho = s+\sqrt{(s^2+1)}$, $\theta = 0$,

$$\Phi \sim \sqrt{(s^2+1)}\,y + isx \quad \text{as} \quad r \to \infty$$
$$= \tfrac{1}{2}(\rho+\rho^{-1})\,y + \tfrac{1}{2}i(\rho-\rho^{-1})\,x.$$

We may move continuously to the required point by the following sequence

$$\begin{aligned}
\Phi &= \tfrac{1}{2}\rho(y+ix) \quad \text{as} \quad r\to\infty, \quad \rho\to\infty, \quad \theta = 0;\\
&\sim \tfrac{1}{2}\rho(\eta+i\xi) \quad \text{as} \quad r\to\infty, \quad \rho\to\infty, \quad \theta \neq 0;\\
&\sim \tfrac{1}{2}\rho(\eta+i\xi) \quad \text{as} \quad \rho\to\infty, \quad \text{finite } r, \quad \theta \neq 0;\\
&\left.\begin{aligned}&\sim \sqrt{(\eta^2-1)}\operatorname{sgn}\eta \quad \text{if} \quad |\eta| > 1,\\ \text{or} \quad &\sim \sqrt{(1-\eta^2)}\,i\operatorname{sgn}\xi \quad \text{if} \quad |\eta| < 1,\end{aligned}\right\} \quad \text{as} \quad \rho\to 1, \quad \text{finite } r, \quad \theta \neq 0.
\end{aligned} \tag{7.7}$$

Only the last stage needs elaboration. If, for $1 < \rho < \infty$, $\Phi = \alpha + i\beta$; then

$$2\alpha\beta = \mathscr{I}[\Phi^2] = \tfrac{1}{2}(\rho^2-\rho^{-2})\,\xi\eta.$$

Thus as ρ decreases from ∞ to 1 neither α nor β pass through zero. But for $\rho = 1$, either α or β vanishes, so the other has the sign it had at $\rho = \infty$. Sgn ξ is $+1$ if ξ is positive, -1 if ξ is negative.

8. The asymptotics for large t

(*a*) *Structure*

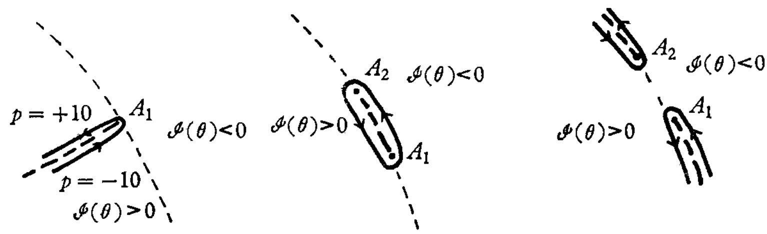
If the singularities on the circle $\rho = 1$ are all distinct, the contour of integration may be deformed to Γ_3, shown in figure 7 (*c*). Then all the contour lies in parts of the ζ-plane where $\mathscr{R}(s) < 0$, except for neighbourhoods around each of the singularities C, C', A_1, A_2 and Q. Except for these neighbourhoods, the contribution to the integral for ψ is exponentially small as $t \to \infty$ and may be ignored. Separate contributions may be evaluated for each singularity, using an approximate form for $\hat{\psi}$, valid in that neighbourhood. The branch cuts connecting A_1, A_2, A_1', A_2' must also be deformed appropriately, the correct branch on $\rho = 1$ being given by equation (7.7).

The structure of the solution in the physical (x, y)-plane at large t is closely bound up with the coincidence or near-coincidence of the singularities on the unit circle in the ζ-plane. For certain critical regions which decrease in size as $t \to \infty$, the contributions cannot be evaluated from the neighbourhood of each singularity separately, and a somewhat more complicated contour of integration must be considered. Herein lies the main analytical advance of this paper. If this is not done, the critical regions appear as singular curves in the limit $t \to \infty$. The method is akin to that of steepest descents (Jeffreys & Jeffreys 1956), but it is not easy to write down general expressions for all the higher order terms in the asymptotic series. However, it is the first dominant term which is mainly of interest, and this is easily obtained by ignoring all but the essential features of the integrand and the contour of integration in the asymptotically small neighbourhoods which contribute to the integral. This lays down the first term and the structure of a power series expansion of the integrand, and higher order corrections come from considering more terms in the series.

Each region in the (x, y)-plane must be considered separately. This entails substantial repetition of similar straightforward calculations. Only key formulae will be given—the interested reader may easily fill in the gaps himself.

(*b*) *Region I*

In region I which covers the whole space $r > 1$ for fixed (x, y) as $t \to \infty$ with the exception of certain lines where the result is singular, the singularities are all distinct. The contour of integration is Γ_3 in figure 7(*c*). The pole at Q is only present in the second problem.



(*a*) (*b*) (*c*)

FIGURE 8. Contours of integration near singularities. (*a*) Region I. (*b*) Region II. (*c*) Region III.

The appropriate form for $\hat{\psi}(\theta)$ in the neighbourhood of A_1 is found by putting

$$\theta = \theta_1 + \lambda, \quad |\lambda| \ll 1, \tag{8.1}$$

and retaining the dependence on λ only in

$$\Phi = (\eta^2 - 1)^{\frac{1}{2}} \sim (-2\xi_1\eta_1\lambda)^{\frac{1}{2}}$$

and in

$$\exp(it\sin\theta) \sim \exp\{it(\sin\theta_1 + \lambda\cos\theta_1)\}.$$

For large t it is only the neighbourhood $|\lambda| t = O(1)$ which contributes to the integral; hence the justification for a power series expansion in λ. This may be seen by putting

$$\lambda = \tfrac{1}{2}i\frac{\mu^2}{t\cos\theta_1}, \tag{8.2}$$

so the exponential becomes $\exp\{it\sin\theta - \frac{1}{2}\mu^2\}$, while the contour of integration is the real axis of μ from large negative values to large positive values (figure 8(*a*)). Following equation (7.7), the correct interpretation of $(-2\xi_1\eta_1\lambda)^{\frac{1}{2}}$ is

$$i\,\mathrm{sgn}\,\eta_1\sqrt{\left(\frac{p}{t\cos\theta_1}\right)}\exp\{i\tfrac{1}{4}\pi\,\mathrm{sgn}\,(\xi_1\eta_1)\}\,\mu.$$

The integration in equation (7.3), now becomes

$$\int_{-\infty}^{+\infty}\exp(-\tfrac{1}{2}\mu^2)\mu^2\,d\mu = \sqrt{(2\pi)},$$

and for the first problem

$$\psi = \mathscr{R}\left[-\frac{X}{\sqrt{(2\pi)}}\mathrm{sgn}\,\eta_1\frac{1}{t}\sqrt{\left(\frac{p}{t\cos\theta_1}\right)}\cos(\theta_1 + \tfrac{1}{4}\pi\,\mathrm{sgn}\,(\xi_1\eta_1) + t\sin\theta_1)\right]. \tag{8.3}$$

This is equation (5.1), with the signs of the quantities ξ_1, η_1 determined by equation (7.6). A formally identical contribution comes from the singularity A_2 with θ_1, ξ_1, η_1 replaced by θ_2, ξ_2, η_2. For the second problem equation (6.1) is established in an identical manner.

Also in region I are contributions from C, C'. Here the standard method of steepest descents (Jeffreys & Jeffreys 1956) may be used, for the integrand in equations (7.4) (7.5) is an analytic function of θ in the neighbourhood of the saddle points C, C' where $d/d\theta\,(\sin\theta) = 0$, and the large parameter t appears only in the exponent. The result is given in equations (5.3, 6.2).

In problem two there is also a pole Q at $\theta = 0$. The contribution from this is simply $2\pi i$ times the residue there. Care is required over the correct branch of $\{\eta^2-1\}^{\frac{1}{2}}$, but the result is equation (6.3).

Region II

As $t \to \infty$ while r/t and ϕ are kept fixed, the singularities A_1, A_2 are separated by an angle $\theta_1 - \theta_2$ of order $1/t$. The relevant part of the contour of integration is now as in figure 8(*b*). If

$$\theta = \phi + \lambda \quad (|\lambda| \ll 1,\ 0 < \phi < \tfrac{1}{2}\pi),$$

and

$$(\eta^2-1)^{\frac{1}{2}} \sim (r^2\lambda^2-1)^{\frac{1}{2}} = i\sqrt{(1-r^2\lambda^2)},$$

along the arc A_1A_2 corresponding to $\mathscr{I}(\theta) \to 0$ from negative values. For values of ϕ in other quadrants the sign may be obtained by symmetry. For the dominant term the only place that the variation with λ of the integrand in equation (7.3) need be considered is in $(\eta^2-1)^{\frac{1}{2}}$ and in the exponent $\exp it(\sin\phi + \lambda\cos\phi)$. The substitution

$$\lambda r = \sin\mu$$

then reduces it to

$$\psi = \mathscr{R}\left[i\frac{X}{2\pi}\frac{\cos\phi}{r}\exp\{i(\phi + t\sin\phi)\}\int_0^{2\pi}\cos^2\mu\exp\{i(t\cos\phi/r)\sin\mu\}\,d\mu\right].$$

On integration by parts the integral becomes one of the standard representations of a Bessel function, and we obtain equation (5.4). Similarly equation (6.6) is derived from (7.5).

Region III

This asymptotic representation arises as $r \to 1$ from the coincidence of A_1 and A_2', i.e.

$$\theta_1 \sim \phi + \tfrac{1}{2}\pi, \quad \theta_2 = \phi - \tfrac{1}{2}\pi.$$

In accordance with our convention that $-\frac{1}{2}\pi < \theta_1, \theta_2 < \frac{1}{2}\pi$, this appears as a coincidence of A_1 and A_2, but the contour of integration is as in figure 8(*c*). Putting

$$\theta = \phi - \tfrac{1}{2}\pi - \lambda \quad (|\lambda| \ll 1,\ 0 \leqslant \phi < \tfrac{1}{2}\pi),$$

then

$$\{\eta^2-1\}^{\frac{1}{2}} = \sqrt{(r^2-1-\lambda^2)}$$

between A_1 and A_2. After the branch has been determined in this way the contour of integration may be deformed with the branch cuts into a small loop around A_1A_2, and as in region II a Bessel function emerges, equations (5.7), (6.7).

Regions IV, V, VI

These arise from the near-coincidence with C of A_1 or A_2, or both. By methods similar to those in regions II and III great simplifications may be made in the integrals (7.3) and (7.5), but these are not enough to enable the result to be expressed in terms of tabulated functions, except for region IV where parabolic cylinder functions emerge. Otherwise two parameters are necessary, but these are of the form described in equations (4.1)–(4.3).

Regions VII, VIII, IX

Here the pole at Q is close to the singularity A_1, the pair A_1, A_2 and the pair A_1, A_2' respectively. The method of evaluation is exactly the same as in regions I, II, III but the contour of integration must be extended to include the pole (figure 9*a*–*c*). That equations (6.10), (6.8), (6.11) are indeed the correct contributions

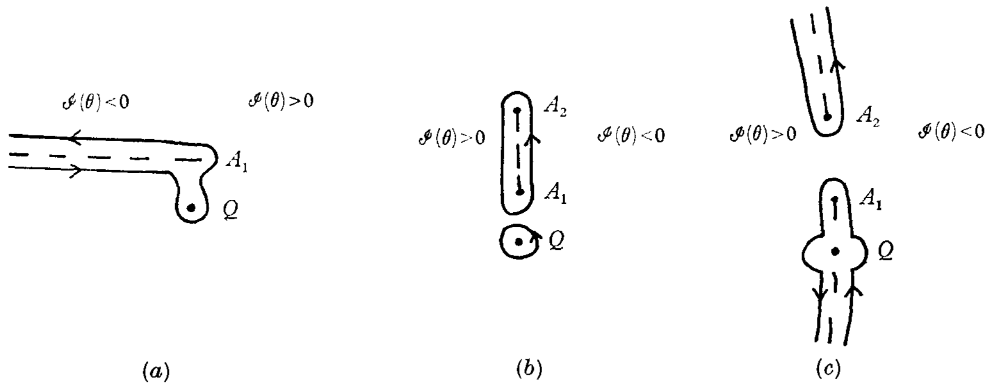


FIGURE 9. Contours of integration near singularities in problem 2. (*a*) Region VII. (*b*) Region VIII. (*c*) Region IX.

follows because on differentiating them with respect to t, they become equivalent to (5.1), (5.4) and (5.7), for the appropriate directions

$$\left(\theta_1 = \frac{y-1}{|x|},\quad \phi = \frac{y}{r},\quad \phi = \frac{\pi}{2}-x\right)$$

and the pole in the simplified integrand of equation (7.5) disappears, to yield the integrals from which (5.1) ,(5.4) and (5.7) were obtained. That the lower limit of integration may be taken as $t = 0$ follows because, although the simplification made in equation (7.5) by taking λ small cannot be justified as an approximation near $t = 0$, the simplified expressions themselves assume definite calculable values at $t = 0$.

9. The effects of viscosity

Although the analysis so far has been entirely inviscid, the insight obtained makes it possible to assess the effect on a Taylor column of a slight viscosity ν'. Allowing for diffusion of vorticity, equation (2.10) becomes

$$\left(\frac{\partial}{\partial t}-\epsilon\nabla^2\right)^2\nabla^2\psi+\frac{\partial^2\psi}{\partial x^2} = 0, \tag{9.1}$$

where $\epsilon = \nu'/2\Omega' a'^2$, and the additional boundary condition must be imposed

$$\partial\psi/\partial r = 0 \quad \text{on} \quad r = 1. \tag{9.2}$$

The problem is still linear, the Laplace transform may still be taken, but there is no longer any simple formal solution corresponding to equation (7.1). However, if ϵ is small, it is possible to proceed heuristically, and break the problem down into two separate parts.

These two parts concern the generation and the propagation of inertial waves respectively. Except for the smallest scales of wavelength comparable with $(\nu'/2\Omega')^{\frac{1}{2}}$, it is to be anticipated that the generation by an impulsive change in the flow at infinity differs little from that for a completely inviscid fluid, and that once generated, such waves will begin to propagate away from the cylinder. Again excepting the smallest scales, the effects of viscosity on plane waves of wave-number α will only be appreciable after the waves have travelled for some time, and are remote from any solid boundaries. Under such conditions they decay exponentially at a rate of $\epsilon\alpha^2$. Thus, while such a wave has travelled a distance p with its appropriate group velocity, the amplitude has decayed by a factor

$$\exp(-\epsilon\alpha^3 p/\cos\phi). \tag{9.3}$$

If $\epsilon\alpha^3/\cos\phi$ is small, this factor is not significantly different from unity unless p is large, and it is thus consistent to compute it in this way, as if no rigid boundaries were present.

Including the factor (9.1) in the integral over wave-number (equation 6.8) which describes the superposition of inertial waves at the nose of a growing Taylor column, we obtain

$$u = U\left\{1 - \int_0^{t/r} \exp(-\epsilon\alpha^3 r)\, J_1(\alpha)\cos\alpha y\, d\alpha\right\}. \tag{9.4}$$

Unlike the inviscid problem, this has a finite limit everywhere as $t \to \infty$,

$$u = U\left\{1 - \int_0^{\infty} \exp(-\epsilon\alpha^3 r)\, J_1(\alpha)\cos\alpha y\, d\alpha\right\}. \tag{9.5}$$

For values of α for which $\epsilon\alpha^3$ is comparable with or larger than unity the integrand in this expression is inaccurate, but provided $r \gg 1$, it is negligibly small for these values, and the total integral is unaffected.

Equation (9.3) shows that in the linear steady-state motion of a slightly viscous fluid at sufficiently low Rossby number, the structure of the Taylor column ahead (or behind) the cylinder closely resembles that for a growing time dependent column in an inviscid fluid. For the factor $\exp(-\epsilon\alpha^3 r)$ differs little from unity for values of the wave-number less than $(\epsilon r)^{-\frac{1}{3}}$, but is very small for values only slightly greater. There is a cut-off wave-number at each point.

Short waves are dissipated before they reach there; longer ones travel faster and are damped slower and are little affected by viscosity. The distance along the column at which the velocity on the axis is $\frac{1}{2}U$ is

$$r = 0{\cdot}3\epsilon^{-1}. \tag{9.6}$$

This computation depends on the generation of inertial waves of wavelength comparable with the cylinder diameter not being significantly affected by viscosity. The waves of wavelength $\epsilon^{\frac{1}{2}}$ or smaller, which appear as an oscillation on the surface of the cylinder cannot be adequately represented by an inviscid theory —the boundary condition (9.2) cannot be ignored. However, these are rapidly damped out and presumably only affect the motion in an Ekman layer near the cylinder.

It is not easy to estimate the magnitude of the non-linear terms which have been ignored in equation (9.1). They are, however, almost certainly negligible for any given value ϵ of the viscosity, provided the Rossby number U is sufficiently small.

Equation (9.3) should be compared with the structure of an axisymmetric wave at large distances from a sphere in a very viscous rotating fluid (for which Stokes flow is a first approximation in the neighbourhood of the sphere). The axial velocity given by Childress (1964, p. 313), when expressed in the present notation is

$$u = U - \tfrac{3}{2} U\epsilon \int_0^\infty \alpha^2 \exp(-\epsilon\alpha^3 r) J_0(\alpha y)\, d\alpha. \tag{9.7}$$

Because the motion is axisymmetric rather than two-dimensional $\cos\alpha y$ in equation (9.5) is replaced by $J_0(\alpha y)$ in (9.7). Childress's solution is valid only for ϵr large, so the cut off wave-number is small. Thus $J_1(\alpha) \sim \frac{1}{2}\alpha$ in equation (9.5) has become $\frac{3}{2}\epsilon\alpha^2$ in (9.7). This change is presumably due in part to the difference in geometry and in part to the difference in character of the source of inertial waves, which is a dipole in the present solution, and a stokeslet in that of Childress.

The dynamics of the column described by equation (9.5) is also quite distinct from that described by Jacobs (1964) when the column is confined between two rigid planes perpendicular to the rotation axis. The freedom of inertial waves to propagate indefinitely until they are dissipated by viscosity is essential to the present picture.

10. Conclusions

The growth of a Taylor column ahead of an obstacle in a rotating fluid may be attributed to the propagation of inertial waves of zero frequency but finite group velocity. The head of the column moves with a constant speed comparable with waves of half-wavelength equal to the diameter of the obstacle. The build-up of singularities at the edge of the column is associated with the successive arrival of waves of smaller and smaller scale. The effects of slight viscosity may easily be included (provided the non-linear terms due to finite Rossby number are even smaller). Small-scale waves are more rapidly damped, and the column is limited to finite length without singularities at the side.

Similar conclusions hold for blocking by a two-dimensional obstacle in a stratified fluid at small internal Froude number. The growth of the region of stagnant fluid ahead and behind is then attributed to internal gravity waves.

The pattern of motion at large times after a transient disturbance may be completely understood in terms of the known group velocity of inertial (internal) waves, with the exception of a distributed non-propagating oscillation at the cut-off frequency.

The persistent oscillation on the surface of the obstacle in an inviscid fluid is described as the residual of inertial (or internal) waves which are of such small scale that they have not yet had time to propagate away.

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